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MIXED CONVECTION FROM A HORIZONTAL CIRCULAR CYLINDER

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Abstract—The combined convection boundary layer on a horizontal circular cylinder in a stream flowing vertically upwards is studied in both the cases of a heated and cooled cylinder. It is found that heating the cylinder delays separation and can, if the cylinder is warm enough, suppress it completely. Cooling the cylinder brings the separation point nearer to the lower stagnation point and for a sufficiently cold cylinder there will not be a boundary layer on the cylinder.

NOMENCLATURE

- a, radius of cylinder;
- g, acceleration of gravity;
- Gr, Grashof number, $= g\beta|\Delta T|a^3/\nu^2$;
- Pr, Prandtl number;
- Q, heat transfer;
- Re, Reynolds number, $= U_0 a/\nu$;
- T, temperature of fluid;
- T_0, temperature of ambient fluid;
- T_1, temperature of cylinder;
- ΔT, temperature difference, $= T_1 - T_0$;
- U_0, free stream;
- x, co-ordinate along surface of cylinder;
- y, co-ordinate normal to cylinder;
- X_s, separation point.

Greek symbols

- α, non-dimensional parameter, $= g\beta\Delta T\ a/U_0^2$;
- β, coefficient of expansion;
- ν, kinematic viscosity;
- τ_w, skin friction.

1. INTRODUCTION

THE PROBLEM of combined convective heat transfer in boundary layers on vertical flat plates for both isothermal and constant heat flux cases has received much attention in the past. Gryzagoridis [1] gives a good description of the previous work on the subject. The problem for a more general body shape has had little discussion. Acrivos [2] considered the form of solution in the limits $Pr \to 0$ and $Pr \to \infty$. Recently Sparrow and Lee [3] looked at the problem of the flow of a vertical stream over a heated horizontal circular cylinder. They obtained a solution by expanding velocity and temperature profiles in powers of x, the co-ordinate measuring distance from the lowest point on the cylinder. This problem is interesting in that both the forced and natural convection boundary layers have the same dependence on x near the lower stagnation point and so both effects are important at the leading edge, whereas for boundary layers on flat plates the natural convection effects modify the forced convection and are not apparent at the leading edge.

In this paper we extend the problem discussed in [3]. We consider the flow of a uniform stream over a horizontal circular cylinder which is held at a constant temperature T_1 surrounded by fluid at temperature T_0, with the stream flowing in the upward vertical direction. For large Reynolds and Grashof numbers, the equations governing the flow are the boundary-layer equations. These are solved numerically using a method similar to that given in [4]. The solution depends on the non-dimensional parameter $\alpha = g\beta\Delta Ta/U_0^2$, where $|\alpha| = Gr/Re^2$. For small $|\alpha|$ forced convection effects dominate, while for large $|\alpha|$ it is the natural convection which is important, so that values of α of $O(1)$, where both effects are comparable, are of most interest. Solutions for small or large α can be obtained as perturbations on the respective forced or natural convection solution.

The cases when $\alpha > 0(T_1 > T_0)$ and $\alpha < 0(T_1 < T_0)$ are considered. For a heated cylinder ($\alpha > 0$) both the forced and natural convection boundary layers start at the lower stagnation point with the buoyancy forces accelerating the fluid in the boundary layer, and so have the effect of reducing the deceleration of the fluid caused by the adverse pressure gradient. In this case the separation of the boundary layer is delayed, and it is found that there is a value of α for which the boundary layer does not separate at all. For values of α greater than this, the boundary layer remains on the cylinder up to the highest point where the boundary layers on each side must collide and leave the surface to form a thin wake above the cylinder. For a cooled cylinder ($\alpha < 0$), the buoyancy forces also retard the fluid and so the separation point is brought nearer to the lower stagnation point. A value of α is found for which the boundary layer separates at this point. For values of α less than this a boundary-layer solution is not possible.

Throughout this paper results are given for $Pr = 1$. Some of the calculations were done for $Pr = 0.72$ (air), but these results were found to differ only slightly from those given.

2. EQUATIONS

On the assumption that $\Delta T/T_0 \ll 1$, $Gr \gg 1$ and $Re \gg 1$ the equations are the incompressible boundary-layer equations; the momentum equation including both a pressure gradient term due to the variation in the main stream and a body force term $g\beta(T-T_0)\sin(x/a)$ (x measuring distance from the lower stagnation point and a being the radius of the cylinder) arising from the buoyancy forces. A uniform stream $\frac{1}{2}U_0$ is flowing vertically upwards over the cylinder, so that the free-stream velocity for the boundary-layer equations is $U_0\sin(x/a)$.

In terms of the non-dimensional variables X, Y, θ and $\bar{\psi}$, where $X = x/a$, $Y = Re^{\frac{1}{2}}y/a$, $\bar{\psi} = \nu Re^{\frac{1}{2}}\psi$ and $\theta = (T-T_0)/\Delta T$ (ψ is the stream function defined in the usual way), the equations are

$$\frac{\partial^3\bar{\psi}}{\partial Y^3}+\sin X\cos X+\alpha\theta\sin X = \frac{\partial\bar{\psi}}{\partial Y}\frac{\partial^2\bar{\psi}}{\partial X\partial Y}-\frac{\partial\bar{\psi}}{\partial X}\frac{\partial^2\bar{\psi}}{\partial Y^2} \quad (1)$$

$$\frac{1}{Pr}\frac{\partial^2\theta}{\partial Y^2} = \frac{\partial\bar{\psi}}{\partial Y}\frac{\partial\theta}{\partial X}-\frac{\partial\bar{\psi}}{\partial X}\frac{\partial\theta}{\partial Y} \quad (2)$$

with boundary conditions

$$\bar{\psi} = \frac{\partial\bar{\psi}}{\partial Y} = 0, \quad \theta = 1 \text{ on } Y = 0 \quad (3)$$

and

$$\frac{\partial\bar{\psi}}{\partial Y}\to\sin X, \quad \theta\to 0 \text{ as } Y\to\infty.$$

Here $\alpha = g\beta\Delta Ta/U_0^2$ is a non-dimensional parameter which describes the relative importance of natural convection to forced convection. This is seen by writing $|\alpha| = Gr/Re^2$.

We have $\alpha > 0$ for a heated cylinder in which case the buoyancy force term is positive and this aids the development of the boundary-layer (acting like a favourable pressure gradient), while for $\alpha < 0$ the cylinder is cooled and the buoyancy forces oppose the development of the boundary layer.

3. SOLUTION

Equations (1) and (2) were solved numerically using a method similar to that described in [4]. In this case $\sin X/X\to 1$ as $X\to 0$ so the appropriate transformation is $\bar{\psi} = Xf(X, Y)$. Equations (1) and (2) then become

$$\frac{\partial^3 f}{\partial Y^3}+\frac{\sin X\cos X}{X}+\alpha\frac{\sin X}{X}\theta+f\frac{\partial^2 f}{\partial Y^2}-\left(\frac{\partial f}{\partial Y}\right)^2 = X\left(\frac{\partial f}{\partial Y}\frac{\partial^2 f}{\partial X\partial Y}-\frac{\partial f}{\partial X}\frac{\partial^2 f}{\partial Y^2}\right) \quad (4)$$

$$\frac{1}{Pr}\frac{\partial^2\theta}{\partial Y^2}+f\frac{\partial\theta}{\partial Y} = X\left(\frac{\partial f}{\partial Y}\frac{\partial\theta}{\partial X}-\frac{\partial f}{\partial X}\frac{\partial\theta}{\partial Y}\right) \quad (5)$$

with boundary conditions

$$f = \frac{\partial f}{\partial Y} = 0, \quad \theta = 1 \text{ on } Y = 0$$
$$\frac{\partial f}{\partial Y}\to\frac{\sin X}{X}, \quad \theta\to 0 \text{ as } Y\to\infty. \quad (6)$$

Equations (4) and (5) were solved using $q = \partial f/\partial Y$ and θ as dependent variables as follows. Derivatives in the X-direction were replaced by differences and all other terms averaged. This gave two ordinary differential equations which were then differenced in the Y-direction and the resulting non-linear algebraic equations solved iteratively by the Newton–Raphson method. The linear algebraic equations arising in the iterative process were solved by Choleski decomposition into upper and lower triangular matrices, so that using the particular forms of the equations, the computer storage could be kept to a minimum. The iterations were repeated until the difference was less than 10^{-6}, and errors in the X-direction were kept small by choosing the step length in this direction so that the difference in two solutions, obtained by covering the interval in one and then two steps was less than 5.10^{-5}. A step length of 0.1 in the Y-direction was found satisfactory for an overall accuracy of four figures. The position where the outer boundary condition was applied had to be varied from $Y = 10$ to $Y = 20$.

From the velocity and temperature profiles thus calculated, heat transfer Q and skin friction τ_w defined by

$$Q = -\frac{a}{\Delta T}Re^{-\frac{1}{2}}\left(\frac{\partial T}{\partial y}\right)_0 = -\left(\frac{\partial\theta}{\partial Y}\right)_0$$

and

$$\tau_w = \frac{a}{U_0}Re^{-\frac{1}{2}}\left(\frac{\partial^2\bar{\psi}}{\partial Y^2}\right)_0 = X\left(\frac{\partial^2 f}{\partial Y^2}\right)_0,$$

were evaluated. Values of Q and τ_w for various α are given in Tables 1 and 2 respectively.

The case $\alpha = 0$ is the forced convection solution and values of τ_w have already been obtained by [5] using the transformation suggested by [6]. This transformation could not be used for this problem as it is singular at $X = \pi$. The results show that increasing α delays separation and that separation can be suppressed completely in $0 \leqslant X \leqslant \pi$ for sufficiently large α. The variation of the separation point X_s with α is given in Fig. 1. The actual value of α which first gives no separation is difficult to determine exactly as it has to be found by successive integrations of the equations, and a further difficulty was encountered as the numerical solution indicated an increase in τ_w and Q very

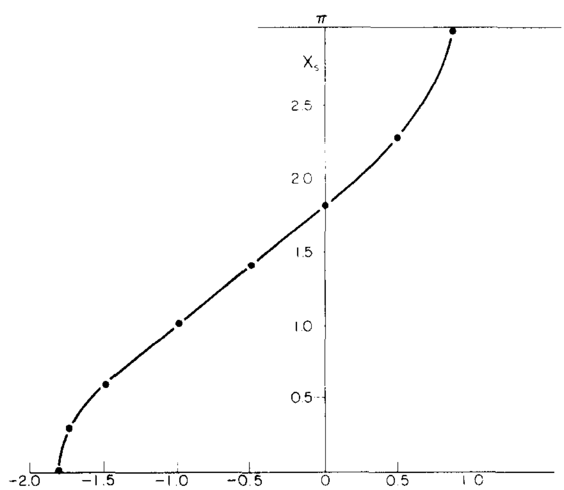

FIG. 1. Variation of the separation point X_s with α.

Table 1. Heat transfer Q for various α

	α										
X	−1.75	−1.5	−1.0	−0.5	0.0	0.5	0.88	0.89	1.0	2.0	5.0
0.0	0.4199	0.4576	0.5067	0.5420	0.5705	0.5943	0.6096	0.6100	0.6156	0.6497	0.7315
0.2	0.4059	0.4498	0.5018	0.5380	0.5668	0.5911	0.6067	0.6071	0.6115	0.6471	0.7261
0.4		0.4236	0.4865	0.5260	0.5564	0.5817	0.5979	0.5983	0.6028	0.6393	0.7193
0.6		0.3373	0.4594	0.5056	0.5391	0.5661	0.5833	0.5837	0.5885	0.6264	0.7082
0.8			0.4160	0.4760	0.5145	0.5443	0.5631	0.5636	0.5686	0.6086	0.6929
1.0			0.3326	0.4353	0.4826	0.5165	0.5375	0.5380	0.5435	0.5863	0.6737
1.2				0.3784	0.4426	0.4828	0.5066	0.5072	0.5133	0.5597	0.6509
1.4				0.2736	0.3928	0.4431	0.4709	0.4716	0.4785	0.5294	0.6248
1.6					0.3280	0.3972	0.4307	0.4314	0.4394	0.4960	0.5959
1.8					0.2114	0.3444	0.3863	0.3872	0.3967	0.4601	0.5645
2.0						0.2821	0.3383	0.3394	0.3509	0.4225	0.5311
2.2						0.1970	0.2871	0.2885	0.3029	0.3842	0.4959
2.4							0.2331	0.2350	0.2540	0.3460	0.4592
2.6							0.1766	0.1796	0.2061	0.3088	0.4205
2.8							0.1162	0.1227	0.1634	0.2730	0.3790
3.0								0.0745	0.1354	0.2381	0.3321
π								0.1033	0.1306	0.2122	0.2918

Table 2. Skin friction τ_w for various α

	α										
X	−1.75	−1.5	−1.0	−0.5	0.0	0.5	0.88	0.89	1.0	2.0	5.0
0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.2	0.0066	0.0533	0.1257	0.1871	0.2427	0.2945	0.3321	0.3330	0.3436	0.4354	0.6803
0.4		0.0741	0.2266	0.3511	0.4627	0.5662	0.6409	0.6429	0.6639	0.8464	1.3318
0.6		0.0026	0.2784	0.4706	0.6393	0.7941	0.9057	0.9085	0.9398	1.2106	1.9277
0.8			0.2554	0.5271	0.7552	0.9614	1.1088	1.1125	1.1538	1.5094	2.4447
1.0			0.1069	0.5051	0.7982	1.0561	1.2383	1.2430	1.2938	1.7295	2.8648
1.2				0.3890	0.7615	1.0727	1.2886	1.2941	1.3541	1.8637	3.1761
1.4				0.1253	0.6429	1.0121	1.2608	1.2671	1.3356	1.9117	3.3729
1.6					0.4405	0.8814	1.1625	1.1695	1.2459	1.8793	3.4557
1.8					0.1069	0.6927	1.0072	1.0491	1.0986	1.7781	3.4300
2.0						0.4599	0.8131	0.8295	0.9117	1.6236	3.3053
2.2						0.1842	0.6012	0.6103	0.7063	1.4334	3.0928
2.4							0.3936	0.4033	0.5048	1.2248	2.8033
2.6							0.2112	p.2219	0.3287	1.0123	2.4447
2.8							0.0711	0.0847	0.1979	0.8043	2.0188
3.0								0.0149	0.1292	0.6002	1.5154
π								0.0504	0.1206	0.4508	1.0919

close to $X = \pi$ for solutions with α near this value. This was thought to be caused by the finite difference replacement of the pressure gradient term, which could be done in several ways. Various forms were tried, all of which gave the above effect with the results in good agreement with each other. The numerical solutions indicate that this value lies between $\alpha = 0.88$ and $\alpha = 0.89$. In fact we can argue that separation will not occur for $\alpha > 1$, as follows. Equation (1) gives, on $Y = 0$,

$$\frac{\partial^3 \bar{\psi}}{\partial Y^3} + \sin X(\alpha + \cos X) = 0.$$

Though $(\partial^2 \bar{\psi}/\partial Y^2)_0 = 0$ at $X = X_s$, the streamwise velocity component $\partial\psi/\partial Y$ will be positive in the neighbourhood of $Y = 0$ and so $(\partial^3 \bar{\psi}/\partial Y^3)_0 \geqslant 0$ at $X = X_s$. From the above, this means that $\sin X(\alpha + \cos X) \leqslant 0$, which cannot hold in $0 \leqslant X \leqslant \pi$ for $\alpha > 1$. The numerical results also show that, in those cases when the boundary layer separates, $\tau_w \to 0$ and $Q \to Q_s(\neq 0)$ as $X \to X_s$ in a singular way as was previously found by [4] for a vertical flat plate.

From Fig. 1, it can be seen that there is a value of $\alpha = \alpha_0$ below which a boundary-layer solution is not possible. The reason is that for $\alpha < 0$ the cylinder is cooled and the natural convection boundary layer would start at $X = \pi$ and for sufficiently small α there comes a point where the flow of the stream upwards cannot overcome the tendency of the fluid next to the cylinder to move downwards under the action of the buoyancy forces. This is an unstable situation and whether a boundary layer can exist at all on the cylinder for $\alpha < \alpha_0$ is still an unanswered question.

The equations for $f_0(Y)$ and $\theta_0(Y)$, the values of f and θ at the stagnation point $X = 0$ are

$$f_0''' + f_0 f_0'' - f_0'^2 + 1 + \alpha\theta_0 = 0 \tag{7}$$

$$\theta_0'' + Pr f_0 \theta_0' = 0 \tag{8}$$

with

$$f_0(0) = f_0'(0) = 0, \quad \theta_0(0) = 1$$
$$f_0' \to 1, \quad \theta_0 \to 0 \text{ as } Y \to \infty, \tag{9}$$

(where primes denote differentiation with respect to Y). These equations are in agreement with those given in [3], and they show that the forced and natural convection effects have the same importance near $X = 0$.

Equations (7) and (8) were solved using the same method as was used in the solution of the ordinary differential equations arising in the solution of the full equations. Values of $f_0''(0)$ and $\theta_0(0)$ for various α are given in Table 3. α_0 was found by solving equations

Table 3. Values of $f_0''(0)$ and $\theta_0'(0)$ for various α

α	$f_0''(0)$	$\theta_0'(0)$	$f_0''(0)$ (series)	$\theta_0'(0)$ (series)
−2.0	−0.29856	−0.33098		
−1.9	−0.09987	−0.38467		
−1.8	0.01950	−0.40993		
−1.6	0.20923	−0.44409		
−1.4	0.36982	−0.46907		
−1.2	0.51460	−0.48935		
−1.0	0.64886	−0.50667		
−0.8	0.77554	−0.52193		
−0.6	0.89627	−0.53566		
−0.4	1.01219	−0.54818		
−0.2	1.12410	−0.55973		
0.0	1.23259	−0.57047		
0.2	1.33810	−0.58052		
0.4	1.44100	−0.58999		
0.6	1.54158	−0.59895		
0.8	1.64007	−0.60747		
1.0	1.73666	−0.61559	1.848	−0.835
1.4	1.92481	−0.63079	2.003	−0.786
1.8	2.10711	−0.64484	2.167	−0.765
2.2	2.28432	−0.65792	2.332	−0.755
2.6	2.45704	−0.67018	2.496	−0.751
3.0	2.62587	−0.68173	2.659	−0.751
4.0	3.03319	−0.70806	3.056	−0.759
5.0	3.42296	−0.73151	3.441	−0.771
6.0	3.79838	−0.75274	3.812	−0.785
7.0	4.16176	−0.77218	4.173	−0.800
8.0	4.51480	−0.79017	4.525	−0.814
9.0	4.84881	−0.80693	4.867	−0.828
10.0	5.19484	−0.82264	5.202	−0.841

(7) and (8) by a matching method subject to the extra boundary condition that $f_0''(0) = 0$, treating α as unknown. This gave $\alpha_0 = -1.81776$. A solution was found for $\alpha < \alpha_0$ which gave $f_0''(0) < 0$.

For $\alpha > 0$, a solution of equations (7) and (8) for large α can be found. Following [2], we make the transformation

$$f_0(Y) = \alpha^{\frac{1}{4}}\phi(\eta), \quad \theta_0(Y) = \theta(\eta) \text{ and } \eta = \alpha^{\frac{1}{4}}Y.$$

The equations become

$$\phi''' + \theta + \phi\phi'' - \phi'^2 + \alpha^{-1} = 0 \tag{10}$$

$$\theta'' + Pr\phi\theta' = 0 \tag{11}$$

with boundary conditions

$$\phi(0) = \phi'(0) = 0, \quad \theta(0) = 1$$
$$\phi' \to \alpha^{-\frac{1}{2}}, \quad \theta \to 0 \text{ as } \eta \to \infty, \tag{12}$$

(where primes denote differentiation with respect to η). (12) suggests an expansion in the form

$$\phi = \phi_0(\eta) + \alpha^{-\frac{1}{2}}\phi_1(\eta) + \alpha^{-1}\phi_2(\eta) + \ldots$$
$$\theta = \theta_0(\eta) + \alpha^{-\frac{1}{2}}\theta_1(\eta) + \alpha^{-1}\theta_2(\eta) + \ldots.$$

The equations for ϕ_0 and θ_0 are those for the natural convection boundary layer at the lower stagnation point of a horizontal cylinder and are given by [7]. The equations for ϕ_i and θ_i $(i = 1, 2, \ldots)$ are linear, and on solving these, expansions for $f_0''(0)$ and $\theta_0'(0)$, valid for large α, are found as

$$f_0''(0) = \alpha^{\frac{3}{4}}(0.81701 + 0.02319\alpha^{-\frac{1}{2}} + 1.00801\alpha^{-1} + \ldots) \tag{13}$$

$$\theta_0'(0) = -\alpha^{\frac{1}{4}}(0.42143 + 0.04701\alpha^{-\frac{1}{2}} + 0.36652\alpha^{-1} + \ldots) \tag{14}$$

Values of $f_0''(0)$ and $\theta_0'(0)$ evaluated from (13) and (14) are given in Table 3, and these show good agreement with the exact values even at moderate values of α.

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CONVECTION MIXTE DEPUIS UN CYLINDRE CIRCULAIRE HORIZONTAL

Résumé—Le couche limite de convection mixte sur un cylindre circulaire horizontal, placé dans un écoulement ascendant vertical, est étudié dans les deux cas du cylindre chauffé et refroidi. On trouve que le chauffage retarde la séparation et peut, si le cylindre est suffisamment chauffé, la supprimer complètement. Le refroidissement du cylindre rapproche le point de séparation du point d'arrêt le plus bas et pour un cylindre assez refroidi il n'y a pas de couche limite sur le cylindre.

GEMISCHTE KONVEKTION AN EINEM HORIZONTALEN KREISZYLINDER

Zusammenfassung—Sowohl für beheizte wie für gekühlte, horizontale Kreiszylinder wird die Grenzschicht bei gemischter Konvektion und senkrecht aufsteigender Strömung untersucht. Es wurde festgestellt, daß durch die Beheizung des Zylinders die Ablösung verzögert wird; bei entsprechend hohen Zylindertemperaturen tritt überhaupt keine Ablösung mehr auf. Eine Kühlung des Zylinders bewirkt eine Annäherung des Ablösepunktes an den unteren Staupunkt. Bei genügend tiefen Zylindertemperaturen ist überhaupt keine Grenzschicht mehr am Zylinder zu beobachten.

СМЕШАННАЯ КОНВЕКЦИЯ ОКОЛО ГОРИЗОНТАЛЬНОГО КРУГЛОГО ЦИЛИНДРА

Аннотация — Исследуется пограничный слой на поверхности горизонтального круглого цилиндра (при нагреве и охлаждении) при смешанной конвекции в направленном вертикально вверх потоке. Показано, что нагрев цилиндра затягивает отрыв пограничного слоя и даже может полностью исключить его в случае сильного нагрева. Охлаждение цилиндра смещает точку отрыва пограничного слоя к критической точке, а в случае достаточного охлаждения цилиндра пограничный слой не образуется на его поверхности.